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Pfaffianization of the discrete KP equation

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Abstract
We use the procedure of Ohta and Hirota to generate an integrable, coupled system of discrete equations from the discrete KP equation.

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1. Introduction

In the early 1990s Ohta and Hirota [1] introduced a procedure for generalizing equations from the KP hierarchy to produce coupled systems of equations with solutions in the form of Pfaffians [2]. Like the KP equations that produced them, these 'Pfaffianized' coupled equations are integrable and have soliton solutions. The majority of integrable nonlinear equations can be written in bilinear form and the solutions of these bilinear equations can usually be expressed as determinants. Pfaffians have a richer structure than determinants and hence the process of Pfaffianization can provide new systems of integrable equations with a broader class of solutions than those having determinantal solutions.

The KP hierarchy in bilinear form [3–5] has solutions in the form of τ-functions. These can be written as Wronskian or Grammian determinants. Ohta and Hirota replaced the Wronskians and Grammians in the bilinear equations with Pfaffians. The resulting equations are only satisfied by the introduction of additional terms.

In their nonlinear form, the equations form a new hierarchy of coupled KP equations. These resulting equations have a richer structure than the original KP equations. One reason for this is that Pfaffian bilinear identities contain more terms than the corresponding Wronskian bilinear identities. A second reason is that the Pfaffianization process introduces new fields into the equations.

In this paper we extend the method of Pfaffianization to discrete equations. We derive the Pfaffianized form of the discrete KP (dKP) equation. This generates an integrable, coupled discrete system of equations.

2. The discrete KP equation (dKP)

The dKP equation [6,7] is the discrete analogue of the more familiar KP equation. The equation involves three independent variables all of the same weight and can be written in bilinear form as

$$\begin{aligned} a_1(a_2-a_3)\tau(k_1+a_1,k_2,k_3)\tau(k_1,k_2+a_2,k_3+a_3) \\ +a_2(a_3-a_1)\tau(k_1,k_2+a_2,k_3)\tau(k_1+a_1,k_2,k_3+a_3) \\ +a_3(a_1-a_2)\tau(k_1,k_2,k_3+a_3)\tau(k_1+a_1,k_2+a_2,k_3)=0. \end{aligned} \tag{1}$$

Here a_1, a_2, a_3 are the lattice spacings in the three directions corresponding to k_1, k_2 and k_3, the three discrete independent variables. There is in fact a whole hierarchy of equations. However, here we will only concern ourselves with this first equation.

The solutions to this equation have been given in terms of Casorati determinants [7]. These are a discrete analogue of the Wronskians that solve the corresponding KP equation. Let φ_i be arbitrary functions of the k_i and an integer S satisfying the dispersion relation

$$\Delta_{k_j}\varphi_i(k_1,k_2,k_3,S)=\varphi_i(k_1,k_2,k_3,S+1) \qquad j=1,2,3. \tag{2}$$

Here Δ_k is the backwards difference operator defined by its action on functions of the discrete variables:

$$\Delta_{k_\nu}F(k_\nu)=\frac{F(k_\nu)-F(k_\nu-a_\nu)}{a_\nu}.$$

Ohta *et al* [7] showed that the solution of the dKP equation (1) is given by the Casorati determinant

$$\tau(k_1,k_2,k_3)=\begin{vmatrix} \varphi_1(k_1,k_2,k_3,0) & \varphi_1(k_1,k_2,k_3,1) & \cdots & \varphi_1(k_1,k_2,k_3,N-1) \\ \varphi_2(k_1,k_2,k_3,0) & \varphi_2(k_1,k_2,k_3,1) & \cdots & \varphi_2(k_1,k_2,k_3,N-1) \\ \vdots & \vdots & & \vdots \\ \varphi_N(k_1,k_2,k_3,0) & \varphi_N(k_1,k_2,k_3,1) & \cdots & \varphi_N(k_1,k_2,k_3,N-1) \end{vmatrix}.$$

It is convenient to introduce some more compact notation. If we write

$$\left|S_1\,{}_{\substack{k_{11}\\k_{21}\\k_{31}}},\ S_2\,{}_{\substack{k_{12}\\k_{22}\\k_{32}}},\ \ldots,\ S_N\,{}_{\substack{k_{1N}\\k_{2N}\\k_{3N}}}\right|$$

$$=\begin{vmatrix} \varphi_1(k_{11},k_{21},k_{31},S_1) & \varphi_1(k_{12},k_{22},k_{32},S_2) & \cdots & \varphi_1(k_{1N},k_{2N},k_{3N},S_N) \\ \varphi_2(k_{11},k_{21},k_{31},S_1) & \varphi_2(k_{12},k_{22},k_{32},S_2) & \cdots & \varphi_2(k_{1N},k_{2N},k_{3N},S_N) \\ \vdots & \vdots & & \vdots \\ \varphi_N(k_{11},k_{21},k_{31},S_1) & \varphi_N(k_{12},k_{22},k_{32},S_2) & \cdots & \varphi_N(k_{1N},k_{2N},k_{3N},S_N) \end{vmatrix}$$

then the solution, τ, becomes

$$\tau(k_1,k_2,k_3)=\left|0\,{}_{\substack{k_1\\k_2\\k_3}},\ 1\,{}_{\substack{k_1\\k_2\\k_3}},\ \ldots,\ N-1\,{}_{\substack{k_1\\k_2\\k_3}}\right|.$$

We can make a further notational simplification by suppressing the indices where the variables are unshifted to leave

$$\tau=|0,1,2,3,\ldots,N-1|.$$

We can obtain expressions for the shifted τ by use of the dispersion relation (2). For instance

$$\tau(k_1+a_1)=|0,1,\ldots,N-2,N-1_{k_1+a_1}|$$

$$\tau(k_1+a_1,k_2+a_2)=\frac{1}{a_1-a_2}\,|0,1,\ldots,N-3,N-2_{k_2+a_2},N-2_{k_1+a_1}|$$

$$\tau(k_1+a_1,k_2+a_2,k_3+a_3) = \frac{1}{(a_1-a_2)(a_1-a_3)(a_2-a_3)} \times|0,1,\ldots,N-4,N-3_{k_3+a_3},N-3_{k_2+a_2},N-3_{k_1+a_1}|.$$

It can be shown that the Casorati determinant is a solution of the dKP by means of a Laplace expansion of determinants [8]. The details of this proof can be found in [7].

3. Properties of Pfaffians

A Pfaffian is the square root of an even-sized skew-symmetric matrix and consequently the properties of Pfaffians are closely related to those of determinants. A Pfaffian is written as a triangular array of elements $A = (a_{ij})$, where $1 \leqslant i < j \leqslant n$. The *Pfaffian* of A, $\mathrm{pf}A$, is defined as follows:

$$\mathrm{pf}A = \left| \begin{matrix} a_{12} & a_{13} & \cdots & a_{1n} \\ & a_{23} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{n-1n} \end{matrix} \right| = \sum_{\sigma} \mathrm{sgn}(\sigma)\, a_{i_1i_2}a_{i_3i_4}\cdots a_{i_{n-1}i_n}$$

where the summation is taken over all permutations $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}$ satisfying

$$i_1 < i_2,\; i_3 < i_4, \ldots,\; i_{n-1} < i_n \qquad \text{and} \qquad i_1 < i_3 < \cdots < i_{n-1}$$

and $\mathrm{sgn}(\sigma) = \pm 1$ denotes the parity of the permutation σ. Note that n must be even. For example, when $n = 2$ we have $\mathrm{pf}A = a_{12}$ and when $n = 4$

$$\mathrm{pf}A = \left| \begin{matrix} a_{12} & a_{13} & a_{14} \\ & a_{23} & a_{24} \\ & & a_{34} \end{matrix} \right| = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

The a_{ij} is taken to be skew-symmetric so that

$$a_{ji} = -a_{ij}.$$

We can introduce a more compact notation by considering a set of labels $\alpha_1, \ldots, \alpha_m$ and a skew-symmetric pairing $(\cdot,\cdot)$. Then we can write

$$\mathrm{pf}(\alpha_1,\alpha_2,\ldots,\alpha_m) := \left| \begin{matrix} (\alpha_1,\alpha_2) & (\alpha_1,\alpha_3) & \cdots & (\alpha_1,\alpha_m) \\ & (\alpha_2,\alpha_3) & \cdots & (\alpha_2,\alpha_m) \\ & & \ddots & \vdots \\ & & & (\alpha_{m-1},\alpha_m) \end{matrix} \right|.$$

For example, we may write $\mathrm{pf}A = \mathrm{pf}(1,2,\ldots,n)$ where $(i,j) := a_{ij}$.

4. Pfaffianizing the KP equation

Armed with some properties of Pfaffians we shall briefly review the Pfaffianization process for the KP [1]. In bilinear form the KP equation takes the form [5]

$$(D_1^4 - 4D_1D_3 + 3D_2^2)\tau\cdot\tau = 0 \tag{3}$$

where the $D_i = D_{x_i}$ are standard Hirota derivatives defined by

$$D_1^m D_2^n D_3^p f\cdot g = (\partial_{x_1} - \partial_{x_1'})^m(\partial_{x_2} - \partial_{x_2'})^n(\partial_{x_3} - \partial_{x_3'})^p f(x_1,x_2,x_3) \times g(x_1',x_2',x_3')\big|_{x_1'=x_1,x_2'=x_2,x_3'=x_3}.$$

The soliton solution to equation (3) takes the form of an $N \times N$ Wronskian determinant

$$\tau = \begin{vmatrix} f_1 & f_1^{(1)} & \cdots & f_1^{(N-1)} \\ f_2 & f_2^{(1)} & \cdots & f_2^{(N-1)} \\ \vdots & & \ddots & \vdots \\ f_N & f_N^{(1)} & & f_N^{(N-1)} \end{vmatrix}$$

where the superscript (n) denotes n derivatives with respect to x_1. Suppressing explicit reference to the functions allows us to write this as

$$\tau = (0, 1, 2, 3, \ldots, N-1).$$

The KP equation in bilinear form can be thought of as a determinantal identity,

$$(\widehat{N-3}, N, N+1)(\widehat{N-1}) - (\widehat{N-3}, N-1, N+1)(\widehat{N-2}, N) \\ +(\widehat{N-2}, N+1)(\widehat{N-3}, N-1, N) = 0 \tag{4}$$

where $(\hat{M}) = (0, 1, 2, \ldots, M)$. If we now replace our Wronskians with Pfaffians we will obtain a similar (but not exactly the same) identity

$$\mathrm{pf}(\widehat{N-3}, N, N+1)\mathrm{pf}(\widehat{N-1}) - \mathrm{pf}(\widehat{N-3}, N-1, N+1)\mathrm{pf}(\widehat{N-2}, N) \\ +\mathrm{pf}(\widehat{N-2}, N+1)\mathrm{pf}(\widehat{N-3}, N-1, N) = \mathrm{pf}(\widehat{N-3})\mathrm{pf}(\widehat{N+1}).$$

Here, as is typically the case, the simplest three-term determinantal identity has been replaced by a four-term Pfaffian identity. The elements in these Pfaffians obey dispersion relations of the form

$$\frac{\partial}{\partial x_n}\mathrm{pf}(l, m) = \mathrm{pf}(l+n, m) + \mathrm{pf}(l, m+n).$$

This kind of dispersion relation is typical for Pfaffians, and the resulting differential rule for a Pfaffian is the same as the differential rule for the Wronskian: for example

$$\frac{\partial}{\partial x}\mathrm{pf}(\widehat{m-1}) = \mathrm{pf}(\widehat{m-2}, m)$$

$$\frac{\partial^2}{\partial x^2}\mathrm{pf}(\widehat{m-1}) = \mathrm{pf}(\widehat{m-2}, m+1) + \mathrm{pf}(\widehat{m-3}, m-1, m)$$

as compared with

$$\frac{\partial}{\partial x}(\widehat{m-1}) = (\widehat{m-2}, m)$$

$$\frac{\partial^2}{\partial x^2}(\widehat{m-1}) = (\widehat{m-2}, m+1) + (\widehat{m-3}, m-1, m)$$

for the Wronskian.

These Pfaffians no longer satisfy the bilinear form of the KP equation (3), but, instead, satisfy the equation

$$(D_1^4 - 4D_1D_3 + 3D_2^2)F \cdot F = 24G\tilde{G} \tag{5}$$

where $F = \mathrm{pf}(\widehat{N-1})$, $G = \mathrm{pf}(\widehat{N+1})$ and $\tilde{G} = \mathrm{pf}(\widehat{N-3})$. This process of Pfaffianization has introduced two new fields $\tilde{G}$ and G into the system and hence additional equations are required to close the system. These can be obtained by looking at further Pfaffian identities. This leads to further bilinear equations

$$(D_1^3 + 2D_3 + 3D_1D_2)\tilde{G} \cdot F = 0 \tag{6}$$

$$(D_1^3 + 2D_3 - 3D_1D_2)G \cdot F = 0. \tag{7}$$

These equations (5)–(7) represent the Pfaffianized KP system. In the next section we shall apply this Pfaffianization process to the dKP equation.

5. Pfaffianizing the dKP equation

In order to Pfaffianize the dKP we require a Pfaffian with elements satisfying the Pfaffianized form of the dispersion relation (2). Hence our entries in our Pfaffian are chosen to satisfy

$$\Delta_{k_n}\mathrm{pf}(i,j) = +\mathrm{pf}(i+1,j) + \mathrm{pf}(i,j+1) - a_n\,\mathrm{pf}(i+1,j+1)$$

or equivalently

$$\mathrm{pf}(i,j)_{k_n-a_n} = \mathrm{pf}(i,j) - a_n\,\mathrm{pf}(i+1,j) - a_n\,\mathrm{pf}(i,j+1) + a_n^2\,\mathrm{pf}(i+1,j+1).$$

In much the same way that the continuous system gives simple expressions for the derivatives of τ-functions, the above discrete dispersion relation gives simple expressions for the shifted τ-functions. Thus if we take

$$\tau = \tau(k_1,k_2,k_3) = \mathrm{pf}(1,2,3,\ldots,N) \qquad N \text{ even}$$

together with $\mathrm{pf}(i,c_j) = a_j^{N+1-i}$ and $\mathrm{pf}(c_i,c_j) = 0$ for $i \neq j$, then we can write down our backward-shifted τ functions as Pfaffians with extra rows

$$\tau_1 = \tau(k_1-a_1,k_2,k_3) = \mathrm{pf}(1,\ldots,N+1,c_1)$$
$$\tau_2 = \tau(k_1,k_2-a_2,k_3) = \mathrm{pf}(1,\ldots,N+1,c_2)$$
$$\tau_3 = \tau(k_1,k_2,k_3-a_3) = \mathrm{pf}(1,\ldots,N+1,c_3)$$
$$\tau_{12} = \tau(k_1-a_1,k_2-a_2,k_3) = \frac{a_1a_2}{a_2-a_1}\mathrm{pf}(1,\ldots,N+1,N+2,c_1,c_2)$$
$$\tau_{13} = \tau(k_1-a_1,k_2,k_3-a_3) = \frac{a_1a_3}{a_3-a_1}\mathrm{pf}(1,\ldots,N+1,N+2,c_1,c_3)$$
$$\tau_{23} = \tau(k_1,k_2-a_2,k_3-a_3) = \frac{a_2a_3}{a_3-a_2}\mathrm{pf}(1,\ldots,N+1,N+2,c_2,c_3)$$
$$\tau_{123} = \tau(k_1-a_1,k_2-a_2,k_3-a_3)$$
$$= \frac{a_1^2a_2^2a_3^2}{(a_2-a_1)(a_1-a_3)(a_3-a_2)}\mathrm{pf}(1,\ldots,N+1,N+2,N+3,c_1,c_2,c_3).$$

Our approach now is to look at Pfaffian identities similar to (but again, not the same as) the determinant identities used in the dKP. There are two simple Pfaffian bilinear identities. They take the form

$$\begin{aligned}&\mathrm{pf}(p_1,p_2,\ldots,p_n,\alpha,\beta,\gamma,\delta)\mathrm{pf}(p_1,p_2,\ldots,p_n)\\ &\quad-\mathrm{pf}(p_1,p_2,\ldots,p_n,\alpha,\beta)\mathrm{pf}(p_1,p_2,\ldots,p_n,\gamma,\delta)\\ &\quad+\mathrm{pf}(p_1,p_2,\ldots,p_n,\alpha,\gamma)\mathrm{pf}(p_1,p_2,\ldots,p_n,\beta,\delta)\\ &\quad-\mathrm{pf}(p_1,p_2,\ldots,p_n,\alpha,\delta)\mathrm{pf}(p_1,p_2,\ldots,p_n,\beta,\gamma) = 0 \end{aligned} \tag{8}$$

and

$$\begin{aligned}&\mathrm{pf}(p_1,p_2,\ldots,p_n,\alpha,\beta,\gamma)\mathrm{pf}(p_1,p_2,\ldots,p_n,\delta)\\ &\quad-\mathrm{pf}(p_1,p_2,\ldots,p_n,\alpha,\beta,\delta)\mathrm{pf}(p_1,p_2,\ldots,p_n,\gamma)\\ &\quad+\mathrm{pf}(p_1,p_2,\ldots,p_n,\alpha,\gamma,\delta)\mathrm{pf}(p_1,p_2,\ldots,p_n,\beta)\\ &\quad-\mathrm{pf}(p_1,p_2,\ldots,p_n,\beta,\gamma,\delta)\mathrm{pf}(p_1,p_2,\ldots,p_n,\alpha) = 0. \end{aligned} \tag{9}$$

The first of these requires n to be even and the second n to be odd. Notice that both these identities have four terms rather than the three terms found in the simplest Jacobi identity for determinants. In the case of dKP we require the following identity:

$$\begin{aligned}&\mathrm{pf}(1,2,\ldots,N+1,N+2,c_1,c_2)\mathrm{pf}(1,2,\ldots,N+1,c_3)\\ &\quad-\mathrm{pf}(1,2,\ldots,N+1,N+2,c_1,c_3)\mathrm{pf}(1,2,\ldots,N+1,c_2)\\ &\quad+\mathrm{pf}(1,2,\ldots,N+1,N+2,c_2,c_3)\mathrm{pf}(1,2,\ldots,N+1,c_1)\\ &\quad-\mathrm{pf}(1,2,\ldots,N+1,c_1,c_2,c_3)\mathrm{pf}(1,2,\ldots,N+1,N+2) = 0 \end{aligned} \tag{10}$$

obtained from (9). This leads to the bilinear equation

$$a_1(a_2-a_3)\tau_{23}\tau_1 - a_2(a_3-a_1)\tau_{13}\tau_2 + a_3(a_1-a_2)\tau_{12}\tau_3 \\ +a_1a_2a_3(a_1-a_2)(a_2-a_3)(a_3-a_1)\tilde{\sigma}_{123}\sigma = 0 \tag{11}$$

which is the discrete analogue of (5). Here σ, $\tilde{\sigma}$ are new fields that correspond to G and $\tilde{G}$ in the Pfaffianized KP equation. The fields σ, $\tilde{\sigma}$ are also Pfaffians of the form

$$\sigma = \mathrm{pf}(1, 2, \ldots, N+1, N+2) \qquad \tilde{\sigma} = \mathrm{pf}(1, 2, \ldots, N-2).$$

The introduction of these two new fields σ and $\tilde{\sigma}$ means that our system is no longer closed so, as with the continuous KP equation, we need to look for further identities. The two required identities are obtained from the remaining Pfaffian bilinear identity (8):

$$a_2a_3(a_2-a_3)\sigma_1\tau_{23} + a_3a_1(a_3-a_1)\sigma_2\tau_{13} + a_1a_2(a_1-a_2)\sigma_3\tau_{12} \\ +(a_1-a_2)(a_2-a_3)(a_3-a_1)\sigma\tau_{123} = 0 \tag{12}$$

$$a_2a_3(a_2-a_3)\tau_1\tilde{\sigma}_{23} + a_3a_1(a_3-a_1)\tau_2\tilde{\sigma}_{13} + a_1a_2(a_1-a_2)\tau_3\tilde{\sigma}_{12} \\ +(a_1-a_2)(a_2-a_3)(a_3-a_1)\tau\tilde{\sigma}_{123} = 0. \tag{13}$$

Thus (11)–(13) represent our Pfaffianized dKP system.

If we wish to consider solutions to this system then we need entries in the Pfaffian that are compatible with the dispersion relation (5) introduced earlier:

$$\mathrm{pf}(i, j)_{k_n-a_n} = \mathrm{pf}(i, j) - a_n\,\mathrm{pf}(i+1, j) - a_n\,\mathrm{pf}(i, j+1) + a_n^2\,\mathrm{pf}(i+1, j+1).$$

Our solution is guided by that of the continuous case [1]. This leads us to express the required Pfaffian entries in the form

$$\mathrm{pf}(i, j) = \sum_{m=1}^{M}[f_{2m-1}(i)f_{2m}(j) - f_{2m-1}(j)f_{2m}(i)]$$

where the f_m satisfy the equations

$$\Delta_{k_n} f_m(i) = f_m(i+1)$$

or equivalently

$$f_m(i; k_n) - f_m(i; k_n - a_n) = a_n f_m(i+1; k_n).$$

We leave further discussion of these solutions to a later publication in which we will compare them to solutions of the discrete BKP (dBKP) [9], which also has Pfaffian-type solutions.

6. Discussion and conclusions

In this paper we have applied the technique of Pfaffianization to derive an integrable, coupled system of discrete equations (11)–(13). These equations depend on the lattice spacings a_i. We can, however, rescale the τ-functions to remove the coefficients in the equations using

$$\tau = [a_1(a_2-a_3)]^{-k_2k_3}[-a_2(a_3-a_1)]^{-k_1k_3}[a_3(a_1-a_2)]^{-k_1k_2}\tau_{\mathrm{old}}$$

where τ_{old} are the τ in equations (11)–(13). This rescaling, together with a similar rescaling for σ and $\tilde{\sigma}$, gives us the system

$$\tau_1\tau_{23} - \tau_2\tau_{13} + \tau_3\tau_{12} = \sigma\tilde{\sigma}_{123}$$
$$\sigma_1\tau_{23} - \sigma_2\tau_{13} + \sigma_3\tau_{12} = \sigma\tau_{123}$$
$$\tau_1\tilde{\sigma}_{23} - \tau_2\tilde{\sigma}_{13} + \tau_3\tilde{\sigma}_{12} = \tau\tilde{\sigma}_{123}.$$

It is interesting to compare this system with the original dKP equation and with the dBKP [9, 10]. In rescaled form these are

$$\tau_1\tau_{23} - \tau_2\tau_{13} + \tau_3\tau_{12} = 0$$

and

$$\tau_1\tau_{23} - \tau_2\tau_{13} + \tau_3\tau_{12} = \tau\tau_{123}$$

respectively. It is clear that the Pfaffianized dKP reduces to the dKP on setting σ or $\tilde{\sigma}$ equal to zero. If, however, we set σ and $\tilde{\sigma}$ equal to τ then we recover the dBKP.

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